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On two-dimensional packets of capillary–gravity waves

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The motion of a two-dimensional packet of capillary–gravity waves on water of finite depth is studied. The evolution of a packet is described by two partial differential equations: the nonlinear Schrödinger equation with a forcing term and a linear equation, which is of either elliptic or hyperbolic type depending on whether the group velocity of the capillary–gravity wave is less than or greater than the velocity of long gravity waves. These equations are used to examine the stability of the Stokes capillary–gravity wave train. The analysis reveals the existence of a resonant interaction between a capillary–gravity wave and a long gravity wave. The interaction requires that the liquid depth be small in comparison with the wavelength of the (long) gravity waves and the evolution equations describing the dynamics of this interaction are derived.

1. Introduction

Recently, our understanding of the evolution of weakly nonlinear surface gravity waves has increased greatly. The contributions by Benjamin & Feir (1967) and Whitham (1967, 1974) showed clearly that a nearly monochromatic, finite amplitude gravity wave is unstable to small modulational or side-band perturbations when the product of the wavenumber k and the fluid depth h exceeds 1·363. Following this important discovery, Benney & Newell (1967) and Hasimoto & Ono (1972) derived a single equation describing the long-time evolution of the envelope of a packet of plane finite amplitude gravity waves and showed that this equation contained the above stability criterion implicitly. Their work, together with that of Zakharov & Shabat (1972), also demonstrated that the evolution equation admitted envelope soliton solutions and that these solitons were intimately related to the terminal state of the side-band instability. Clear experimental confirmation and further discussion of these properties were given subsequently by Yuen & Lake (1975). Davey & Stewartson (1974) extended the theoretical basis for these results to discuss two-dimensional wave packets, but neither the analytical solution nor any experimental evidence regarding the evolution or interaction of two-dimensional packets has yet emerged.

In this paper we extend the analysis of Davey & Stewartson to include the effects of capillarity. The evolution equations derived are a convenient basis for

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discussing the stability of a Stokes capillary–gravity wave train for arbitrary kh. The analysis reveals that there are several distinct wavenumber bands in which the wave train is unstable to modulational perturbations. It also shows that two of the stability boundaries are associated with two separate resonance phenomena. The first corresponds to the second-harmonic resonance elucidated by McGoldrick (1970*a*, *b*, 1972) and the second is associated with a resonance between a shallow-water gravity wave and a capillary wave. The latter requires that the group speed of the capillary wave matches the phase velocity of a shallow-water gravity wave. The dynamical evolution equations describing this resonance phenomenon are derived and their solution is discussed.

2. The self-modulation evolution equations

We consider the evolution of a progressive gravity–capillary wave moving on the free surface of a liquid of constant depth h. The undisturbed free surface corresponds to the plane $z = 0$, where z points vertically upwards, and the bottom is located at $z = -h$. The remaining Cartesian co-ordinates x, y are in the plane of the undisturbed free surface, and we choose x to point in the direction of the wave propagation. Since the fluid motion is irrotational, a velocity potential $\phi(x, y, z, t)$ satisfying Laplace's equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad -h < z < \zeta, \tag{2.1}$$

can be defined, where $\zeta(x, y, t)$ denotes the position of the undulating free surface. The boundary conditions for the motion are

$$\phi_z = 0 \quad \text{at} \quad z = -h, \tag{2.2a}$$

$$\phi_z = \zeta_t + \phi_x \zeta_x + \phi_y \zeta_y \quad \text{at} \quad z = \zeta \tag{2.2b}$$

and

$$g\zeta + \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = T\frac{\zeta_{xx}(1+\zeta_y^2) + \zeta_{yy}(1+\zeta_x^2) - 2\zeta_{xy}\zeta_x\zeta_y}{(1+\zeta_x^2+\zeta_y^2)^{\frac{3}{2}}} \quad \text{at} \quad z = \zeta. \tag{2.2c}$$

The parameter T is the ratio of the surface-tension coefficient to the fluid density and g is the gravitational acceleration. We suppose that initially (at $t = 0$) the surface is distorted in the manner

$$\zeta(x, y, t = 0) = \epsilon \frac{i\omega}{g(1+\tilde{T})}\{A(\epsilon x, \epsilon y)\, e^{ikx} - A^* e^{-ikx}\}, \tag{2.3}$$

where $\tilde{T} = k^2T/g$, the asterisk denotes the complex conjugate and ϵ is a non-dimensional parameter measuring the slope of the wavy surface, which has wavelength $2\pi/k$. The envelope $A(\epsilon x, \epsilon y)$ of the surface distortion is allowed to possess a slow spatial variation and the frequency ω is determined uniquely by the value of k through the dispersion relation

$$\omega = \{gk\sigma(1+\tilde{T})\}^{\frac{1}{2}}, \tag{2.4}$$

where $\sigma = \tanh kh$.

We now derive the equations describing the time evolution of A when the motion is only weakly nonlinear ($0 < \epsilon \ll 1$). On this basis we assume that (2.1) and (2.2) have a solution of the form

$$\phi = \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \epsilon^3\phi^{(3)} + \dots, \tag{2.5a}$$

$$\zeta = \epsilon\zeta^{(1)} + \epsilon^2\zeta^{(2)} + \epsilon^3\zeta^{(3)} + \dots. \tag{2.5b}$$

Also, as shown by the earlier work of Benney & Newell (1967) and Davey & Stewartson (1974), it proves convenient to introduce the multiple scales

$$\xi = \epsilon(x - c_g t), \quad \eta = \epsilon y, \quad \tau = \epsilon^2 t, \quad \xi_1 = \epsilon^2(x - c_g t), \dots. \tag{2.6}$$

c_g denotes the group velocity and is given by the relation

$$c_g = \frac{\partial \omega}{\partial k} = c_p \left\{ \frac{\sigma + kh(1-\sigma^2)}{2\sigma} + \frac{\tilde{T}}{1+\tilde{T}} \right\}, \quad c_p = \frac{\omega}{k}. \tag{2.7}$$

Substituting these forms into the governing set of equations, solving successively the equations resulting from repeated use of the limit process $\epsilon \to 0$, with x, y, z, t, ξ, η and τ fixed, and using the notation

$$E \equiv \exp\{i(kx - \omega t)\}, \tag{2.8}$$

one obtains the following results:

$$\phi^{(1)} = \Phi^{(1,0)}(\xi,\eta,\tau) + \frac{\cosh k(z+h)}{\cosh kh} \{A(\xi,\eta,\tau)E + A^*E^{-1}\}, \tag{2.9a}$$

$$g\zeta^{(1)} = 0 + \frac{i\omega}{1+\tilde{T}} \{AE - A^*E^{-1}\}; \tag{2.9b}$$

$$\begin{aligned} \phi^{(2)} = {} & \Phi^{(2,0)}(\xi,\eta,\tau) + \frac{\cosh k(z+h)}{\cosh kh} \{D(\xi,\eta,\tau)E + D^*E^{-1}\} \\ & - i\,\frac{(z+h)\sinh k(z+h) - h\sigma\cosh k(z+h)}{\cosh kh} \{A_\xi E - A_\xi^* E^{-1}\} \\ & + \frac{3ik^2}{4}\frac{\cosh 2k(z+h)}{\cosh 2kh} \left[\frac{(1+\sigma^2)\{1-\sigma^2+\tilde{T}(3-\sigma^2)\}}{\sigma^2 - \tilde{T}(3-\sigma^2)} \right] \{A^2E^2 - A^{*2}E^{-2}\}, \end{aligned} \tag{2.10a}$$

$$\begin{aligned} g\zeta^{(2)} = {} & c_g \Phi_\xi^{(1,0)} - k^2(1-\sigma^2)|A|^2 + \frac{i\omega}{1+\tilde{T}}\{DE - D^*E^{-1}\} \\ & + \frac{c_p}{1+\tilde{T}} \left[\frac{c_g}{c_p} - \frac{2\tilde{T}}{1+\tilde{T}} \right] \{A_\xi E + A_\xi^* E^{-1}\} \\ & - \frac{k^2}{2} \frac{3-\sigma^2}{\sigma^2 - \tilde{T}(3-\sigma^2)} \{A^2E^2 + A^{*2}E^{-2}\}. \end{aligned} \tag{2.10b}$$

The second-harmonic terms in (2.10) are observed to be singular when $\tilde{T} = \sigma^2/(3-\sigma^2)$, which for deep water ($\sigma = 1$) yields $\tilde{T} = \frac{1}{2}$. Wavenumbers satisfying this condition have the property that the phase speeds of the first and second harmonic match, resulting in the phenomenon known as 'second-harmonic resonance'. The analysis breaks down at this wavenumber and a new scaling is required (cf. McGoldrick 1970*b*).

The significance of the value $\tilde{T} = \frac{1}{2}$ for deep water has been known for some time. Harrison (1909) showed that the influence of nonlinearity on capillary–

gravity waves is quite different depending on whether $\tilde{T}$ is less than or greater than $\frac{1}{2}$. For $\tilde{T} < \frac{1}{2}$ the influence of nonlinearity (higher harmonics) is to distort the wave profile in such a way that the crests are sharpened and the troughs flattened. Profiles of this type are called gravity waves. When $\tilde{T} > \frac{1}{2}$ the influence of nonlinearity is in the opposite sense and is consistent with the results for pure capillary waves given by Crapper (1957). McGoldrick (1970*b*, 1972) gives a complete discussion of the harmonic resonance phenomenon in gravity–capillary waves.

Assuming that the wavenumber k is not too close to that for which $\tilde{T} = \sigma^2/(3-\sigma^2)$, we can continue the analysis to obtain

$$\begin{aligned}\phi^{(3)} = &-\tfrac{1}{2}(z+h)^2\{\Phi^{(1,0)}_{\xi\xi}+\Phi^{(1,0)}_{\eta\eta}\}+\Phi^{(3,0)}(\xi,\eta,\tau)+\frac{\cosh k(z+h)}{\cosh kh}\{G(\xi,\eta,\tau)E+G^*E^{-1}\}\\ &+\frac{(z+h)\sinh k(z+h)-h\sigma\cosh k(z+h)}{2k\cosh kh}\\ &\qquad\times\{(2kh\sigma A_{\xi\xi}-A_{\eta\eta}-2ikD_{\xi}-2ikA_{\xi_1})E+\text{c.c.}\}\\ &-\frac{[(z+h)^2-h^2]\cosh k(z+h)}{2\cosh kh}\{A_{\xi\xi}E+A^*_{\xi\xi}E^{-1}\}\\ &\qquad+\text{higher-harmonic terms.}\end{aligned}\tag{2.11}$$

Then, upon using boundary condition (2.2*b*), we find that the leading-order mean flow or long-wave component is prescribed by the equation

$$(gh-c_g^2)\,\Phi^{(1,0)}_{\xi\xi}+gh\Phi^{(1,0)}_{\eta\eta} = -k^2c_p\left[\frac{c_g}{c_p}(1-\sigma^2)+\frac{2\tilde{T}}{1+\tilde{T}}\right](|A|^2)_{\xi}.\tag{2.12}$$

This equation shows that the long-wave component is generated by the self-interaction of the short wave. Also, upon comparing the first-harmonic terms in (2.2*b*) and (2.2*c*), we find that the two equations are compatible only if $A(\xi,\eta,\tau)$ satisfies the evolution equation

$$\begin{aligned}2i\omega A_{\tau}+\omega\omega''A_{\xi\xi}+c_pc_gA_{\eta\eta} = &\,2k^2c_p\left\{1+\frac{1}{2}\frac{c_g}{c_p}(1-\sigma^2)(1+\tilde{T})\right\}A\,\Phi^{(1,0)}_{\xi}\\ &+\frac{k^4}{2}\left\{\frac{(1-\sigma^2)(9-\sigma^2)+\tilde{T}(3-\sigma^2)(7-\sigma^2)}{\sigma^2-\tilde{T}(3-\sigma^2)}+8\sigma^2-2(1-\sigma^2)^2(1+\tilde{T})-\frac{3\sigma^2\tilde{T}}{1+\tilde{T}}\right\}\\ &\qquad\times|A|^2A.\end{aligned}\tag{2.13}$$

If we introduce a quantity Q defined by

$$Q = \frac{c_g}{k^2}\Phi^{(1,0)}_{\xi}+\frac{c_g}{gh-c_g^2}\left\{\frac{2c_p}{1+\tilde{T}}+c_g(1-\sigma^2)\right\}|A|^2,\tag{2.14}$$

the evolution equations (2.12) and (2.13) can be reduced to a form consistent with that derived by Davey & Stewartson (1974), namely

$$iA_{\tau}+\lambda A_{\xi\xi}+\mu A_{\eta\eta} = \nu|A|^2A+\nu_1AQ\tag{2.15}$$

and

$$(gh-c_g^2)Q_{\xi\xi}+ghQ_{\eta\eta} = \kappa(|A|^2)_{\eta\eta}.\tag{2.16}$$

The coefficients have the following definitions:

$$\left.\begin{aligned}
\lambda &= \tfrac{1}{2}\omega''(k), \quad \mu = \omega'(k)/2k = c_g/2k,\\
\nu &= \frac{k^4}{4\omega}\left\{\frac{(1-\sigma^2)(9-\sigma^2)+\tilde{T}(3-\sigma^2)(7-\sigma^2)}{\sigma^2-\tilde{T}(3-\sigma^2)}+8\sigma^2-\frac{3\sigma^2\tilde{T}}{1+\tilde{T}}\right.\\
&\quad\left.-\frac{8c_g^2}{(gh-c_g^2)(1+\tilde{T})}\left[\left(\frac{c_p}{c_g}\right)^2+\frac{c_p}{c_g}(1-\sigma^2)(1+\tilde{T})+\frac{gh}{c_g^2}(1-\sigma^2)^2(1+\tilde{T})^2\right]\right\},\\
\nu_1 &= \frac{k^4}{\omega}\left[\frac{c_p}{c_g}+\tfrac{1}{2}(1-\sigma^2)(1+\tilde{T})\right],\\
\kappa &= ghc_g\frac{2c_p+c_g(1-\sigma^2)(1+\tilde{T})}{(gh-c_g^2)(1+\tilde{T})}.
\end{aligned}\right\} \quad (2.17)$$

The coefficients reduce to those given by Davey & Stewartson when the surface tension vanishes ($\tilde{T}=0$). However, several important features are present here which are excluded by the limitation to pure gravity waves. First, we note that the equation for Q (or equivalently $\Phi^{(1,0)}$) is of either elliptic or hyperbolic type depending on whether $c_g^2 \lessgtr gh$. For gravity waves of finite wavelength, c_g is always less than $(gh)^{\frac{1}{2}}$ and Q satisfies a Poisson equation. The influence of surface tension is to increase the group velocity and may do so to the extent that c_g can exceed the velocity $(gh)^{\frac{1}{2}}$ of long gravity waves. The equation for Q is then hyperbolic with characteristics

$$\eta = \pm\{(c_g^2/gh)-1\}^{\frac{1}{2}}+\text{constant}. \quad (2.18)$$

One expects that the solutions for A and Q will be quite different in this case when geometrical inhomogeneities, such as depth variations, are present. Second, observe that the coefficient ν of the cubic nonlinear term is singular when $c_g(k) = (gh)^{\frac{1}{2}}$ (it is also singular when the second-harmonic resonance condition is satisfied as noted earlier). This corresponds to a long-wave/short-wave resonance in which the group velocity of the short (capillary) wave matches the phase velocity of the long (gravity) wave. Equations (2.15) and (2.16) break down under this condition and a different analysis and scaling are required. We present the relevant equations describing this resonance in §4.

Except for wavenumbers close to the two resonance conditions $c_g(k) = (gh)^{\frac{1}{2}}$ and $\tilde{T} = \sigma^2/(3-\sigma^2)$, (2.15) and (2.16) describe, to leading order in ϵ, the evolution of a nearly monochromatic, progressive gravity–capillary wave subject to the appropriate initial and boundary conditions. When the wave motion is one-dimensional (i.e. A is independent of η), Q vanishes and (2.15) reduces to the now familiar nonlinear Schrödinger equation, first derived for water waves by Benney & Newell (1967). A method for solving the equation exactly when the influence of the initial conditions vanishes sufficiently rapidly as $|\xi|$ becomes large has been presented by Zakharov & Shabat (1972). Their solution demonstrated the existence of envelope solitons, which, as shown recently by Yuen & Lake (1975), play a key role in the evolution of a deep-water gravity-wave packet and/or gravity-wave train.

Before discussing the stability of wave-train solutions to these equations, we present their form in the deep-water limit ($kh\to\infty$, $\epsilon kh \ll 1$, $\tilde{T}$ fixed). Applying

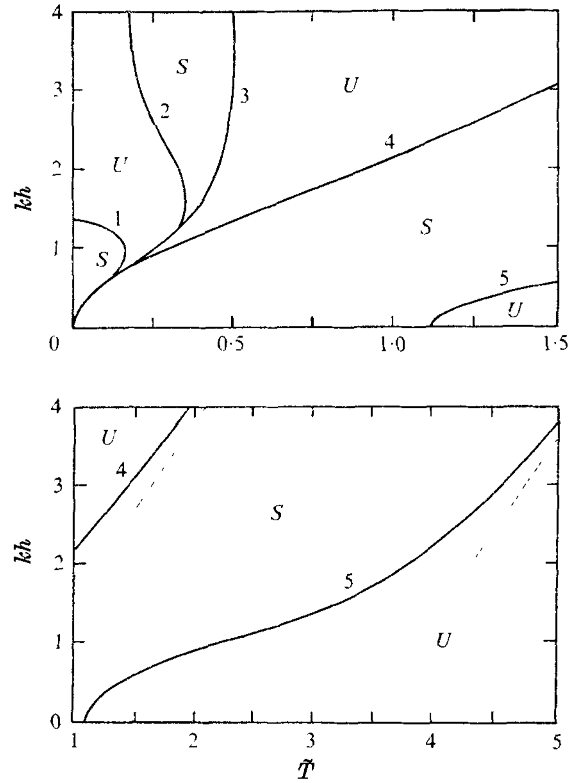

FIGURE 1. Stability diagram for the Stokes capillary–gravity wave train. S and U denote stable and unstable regions. - - - -, asymptotes of the respective curves.

the limit to (2.12), we find that the equation is homogeneous and always of elliptic type. The solution then reduces simply to

$$\Phi_{\xi}^{(1,0)} = \Phi_{\eta}^{(1,0)} = 0, \tag{2.19}$$

and (2.13) becomes

$$2i\omega A_{\tau} - \frac{g}{4k}\frac{1-6\tilde{T}-3\tilde{T}^2}{1+\tilde{T}}A_{\xi\xi} + \frac{g}{2k}(1+3\tilde{T})A_{\eta\eta} = \frac{k^4}{2}\frac{8+\tilde{T}+2\tilde{T}^2}{(1-2\tilde{T})(1+\tilde{T})}|A|^2A. \tag{2.20}$$

The second-harmonic resonance is still present as is manifested by the factor $1-2\tilde{T}$ in the denominator of the nonlinear term, but the long-wave/short-wave resonance has disappeared. We postpone a discussion of this until the next section.

3. Modulational instability of a uniform wave train

Hasimoto & Ono (1972) have shown, for the one-dimensional case, that the evolution equation (2.15) for A is very convenient for examining the stability of a uniform or Stokes wave train. They showed that the wave train is unstable when $\lambda\nu < 0$ and confirmed the result of Benjamin & Feir (1967) that the Stokes gravity wave is unstable when $kh > 1{\cdot}363$. Hayes (1973) and Davey & Stewartson (1974) generalized the stability criterion to the case of two-dimensional waves.

As discussed above, the equation for Q can be hyperbolic when capillarity is included, so that Q is constant along characteristics. However, since there is no reason for Q to change across characteristics in the constant-depth case considered here, the solution $Q = Q_0 =$ constant applies for both $c_g \gtrless (gh)^{\frac{1}{2}}$. Thus the two-dimensional criterion will be unaltered. The stability or instability of a capillary–gravity wave train to oblique disturbances will not be discussed further herein.

The criterion $\lambda\nu < 0$ for instability of the one-dimensional wave train shows that stability boundaries exist for those values of $\tilde{T}$ and kh corresponding to simple zeros of λ and ν, providing they are not coincident, and where ν has a first-order singularity. In figure 1, we show the regions in kh, $\tilde{T}$ space where the Stokes capillary–gravity wave train is modulationally stable and/or unstable. Curves 1 and 5 correspond to simple zeros of the coefficient ν. Curve 1 passes through the point $kh = 1{\cdot}363$ given by Benjamin & Feir (1967). Along curve 2, $\lambda \sim \omega''(k)$ vanishes and the phase velocity has a minimum. Curves 3 and 4 define the resonance cases $\tilde{T} = \sigma^2/(3-\sigma^2)$ and $c_g^2 = gh$ respectively, and ν is singular along each of them. Curve 4 has the asymptote

$$kh = \tfrac{9}{4}\tilde{T} - \tfrac{3}{4}, \quad kh \gg 1, \tag{3.1}$$

and curve 5 has the asymptote

$$kh = \tfrac{9}{4}\tilde{T} - \tfrac{61}{8}, \quad kh \gg 1. \tag{3.2}$$

It is interesting to note that capillary waves with $\tilde{T} \gg 1$ are stable in the region between these two boundaries. Such a result cannot be obtained if the case of pure capillary waves is treated from the beginning [i.e. omitting the term $2g\zeta$ in (2.2*c*)]. In that case one finds that capillary waves are modulationally unstable for all kh.† This apparent inconsistency can be explained in the following way. For $\tilde{T}$ tending to infinity, the terms $\frac{3}{4}$ and $\frac{61}{8}$ in the equations for the asymptotes can be neglected. In this limit the two boundaries coincide, since they have the same inclination, and the stable region between them disappears. Therefore analysis of capillary waves by omitting the effect of gravity can lead to erroneous results regarding the stability of the Stokes wave train over a significant range of wavenumbers.

It is now clear why the resonance condition $c_g^2 = gh$ vanishes in the deep-water limit. Noting that the slope of the resonance locus (curve 4 in figure 1) is finite, the deep-water limit ($kh \to \infty$, $\tilde{T}$ fixed) always places one in the unstable region between curves 3 and 4, providing $\tilde{T} > \frac{1}{2}$ of course. Thus the long-wave/short-wave resonance found herein is uniquely a shallow-water phenomenon so far as the gravity wave (long wave) is concerned.

4. Long wave/short-wave resonant interaction

We now develop the dynamical evolution equations describing the long-wave/short-wave resonant interaction whose existence we noted earlier in §2. The evolution equations (2.15) and (2.16) are singular when $c_g^2 = gh$ and so we expect

† See appendix for details.

that this interaction occurs on a much shorter time scale than that for self-modulation. This indeed turns out to be true, but it is not *a priori* obvious what that time scale should be. The balance we want to achieve must be such that the dispersion of the short wave is balanced by the nonlinear interaction of the long wave with the short wave and such that the evolution of the long wave is driven by the self-interaction of the short wave: a Reynolds-stress-like effect. It emerges that the required time scale is faster than that found by Newell (1977) in his model of a long-wave/short-wave interaction. However, as noted earlier, the resonant interaction considered here vanishes in the deep-water limit.

Keeping in mind the requirements just mentioned, one can show that the expansion for the dependent variables must have the form

$$\phi = \epsilon^{\frac{2}{3}}\phi^{(0)} + \epsilon\phi^{(1)} + \epsilon^{\frac{4}{3}}\phi^{(2)} + \epsilon^{\frac{5}{3}}\phi^{(3)} + \epsilon^{2}\phi^{(4)} + \epsilon^{\frac{7}{3}}\phi^{(5)} + \epsilon^{\frac{8}{3}}\phi^{(6)} + \dots, \tag{4.1}$$

and similarly for ζ. Restricting our discussion to the one-dimensional case, we introduce the multiple scales

$$\xi = \epsilon^{\frac{2}{3}}(x - c_g t), \quad \tau = \epsilon^{\frac{4}{3}}t, \quad \xi_1 = \epsilon^{\frac{4}{3}}(x - c_g t), \dots . \tag{4.2}$$

Although we use the same multiple-scale variables in this section as in the previous ones, but with different definitions, no confusion should arise. In the expansion (4.1), $\phi^{(1)}$ describes the free short-wave mode, so that ϵ is again a non-dimensional measure of the amplitude of that mode. If we then take $\phi^{(0)}$ to describe the long wave a consistent dynamical description is obtained with the result that, to leading order, the long wave and the short wave are free modes of the system.

Using the expansion (4.1) and the multiple-scale variables (4.2) in the governing system of equations (2.1) and (2.2), the following results are obtained:

at $O(\epsilon^{\frac{2}{3}})$,

$$\phi^{(0)} = \Phi^{(0)}(\xi, \tau), \quad \zeta^{(0)} = 0; \tag{4.3}$$

at $O(\epsilon)$,

$$\left.\begin{aligned} \phi^{(1)} &= \Phi^{(1,0)}(\xi,\tau) + \frac{\cosh k(z+h)}{\cosh kh}\{A(\xi,\tau)E + A^*E^{-1}\}, \\ g\zeta^{(1)} &= 0 + i\omega(1+\tilde{T})^{-1}\{AE - A^*E^{-1}\}; \end{aligned}\right\} \tag{4.4}$$

at $O(\epsilon^{\frac{4}{3}})$,

$$\left.\begin{aligned} \phi^{(2)} &= \Phi^{(2,0)}(\xi,\tau) + \frac{\cosh k(z+h)}{\cosh kh}\{F(\xi,\tau)E + F^*E^{-1}\}, \\ g\zeta^{(2)} &= c_g\Phi^{(0)}_{\xi} + i\omega(1+\tilde{T})^{-1}\{FE - F^*E^{-1}\}; \end{aligned}\right\} \tag{4.5}$$

at $O(\epsilon^{\frac{5}{3}})$,

$$\left.\begin{aligned} \phi^{(3)} = {} & \Phi^{(3,0)}(\xi,\tau) + \frac{\cosh k(z+h)}{\cosh kh}\{D(\xi,\tau)E + D^*E^{-1}\} \\ & - i\frac{(z+h)\sinh k(z+h) - h\sigma\cosh k(z+h)}{\cosh kh}\{A_\xi E - A^*_\xi E^{-1}\}, \\ g\zeta^{(3)} = {} & c_g\Phi^{(1,0)}_{\xi} + \frac{i\omega}{1+\tilde{T}}\{DE - D^*E^{-1}\} \\ & + c_p\left(\frac{c_g}{c_p} - \frac{2\tilde{T}}{1+\tilde{T}}\right)\{A_\xi E + A^*_\xi E^{-1}\}; \end{aligned}\right\} \tag{4.6}$$

at $O(\epsilon^2)$,

$$\left.\begin{aligned} \phi^{(4)} &= -\tfrac{1}{2}(z+h)^2\,\Phi^{(0)}_{\xi\xi} + \Phi^{(4,0)}(\xi,\tau) + \frac{\cosh k(z+h)}{\cosh kh}\{H(\xi,\tau)\,E + H^*E^{-1}\} \\ &\quad - i\,\frac{(z+h)\sinh k(z+h) - h\sigma\cosh k(z+h)}{\cosh kh}\{F_\xi E - F^*_\xi E^{-1}\}, \\ g\zeta^{(4)} &= \{c_g(\Phi^{(2,0)}_{\xi} + \Phi^{(0)}_{\xi_1}) - \Phi^{(0)}_{\tau} - k^2(1-\sigma^2)\,|A|^2\} \\ &\quad + \frac{i\omega}{1+\tilde{T}}\{HE - H^*E^{-1}\} + c_p\left[\frac{c_g}{c_p} - \frac{2\tilde{T}}{1+\tilde{T}}\right]\{F_\xi E + F^*_\xi E^{-1}\}. \end{aligned}\right\} \quad (4.7)$$

Applying boundary condition (2.2b) at $O(\epsilon^2)$ shows that $\Phi^{(0)}$ is a free long wave.

Proceeding to the next order we obtain the solution for $\phi^{(5)}$, which is identical to the solution given in (2.11) without the higher-harmonic terms. Then, upon examination of the first-harmonic terms in the boundary conditions (2.2b) and (2.2c) correct to this order, one finds that they are compatible only if the amplitude of the first harmonic satisfies the evolution equation

$$iA_\tau + \lambda A_{\xi\xi} = \delta\Phi^{(0)}_{\xi}A, \quad (4.8)$$

where λ has the same definition as in (2.17) and

$$\delta = k\{1 + \tfrac{1}{2}(c_g/c_p)\,(1-\sigma^2)\,(1+\tilde{T})\}. \quad (4.9)$$

Also, in order to avoid the appearance of secular terms through use of the kinematic boundary condition at $O(\epsilon^{\frac{8}{3}})$, we must have

$$\Phi^{(0)}_{\tau\xi} = -\tfrac{1}{2}k^2(1-\sigma^2)\,(|A|^2)_\xi. \quad (4.10)$$

The above equations imply that the evolution of the long wave is forced by the self-interaction of the short wave and that the short wave is modulated and detuned by its interaction with the long wave.

The evolution equations describing this interaction can be written in a more convenient form using the definition

$$B(\xi,\tau) = \delta\Phi^{(0)}_{\xi}, \quad (4.11)$$

so that we obtain the coupled pair of equations

$$iA_\tau + \lambda A_{\xi\xi} = BA \quad (4.12a)$$

and

$$B_\tau = -\alpha(|A|^2)_\xi. \quad (4.12b)$$

It is expected that these equations will describe resonant interaction in other dispersive phenomena when the coefficient of the nonlinear term in the cubic Schrödinger equation is singular because the group velocity of the modulated wave matches a long-wave phase velocity of the system.† In the present case, the coefficient α is given by

$$\alpha = \tfrac{1}{2}\delta k^2(1-\sigma^2) = \tfrac{1}{2}k^3(1-\sigma^2)\{1 + \tfrac{1}{2}(c_g/c_p)\,(1-\sigma^2)\,(1+\tilde{T})\} > 0 \quad (4.13)$$

and λ, the dispersion of the capillary waves, is also positive.

† While this work was in progress, we discovered that Grimshaw (1975) has derived the same evolution equations in an analysis of internal waves.

The set of equations (4.12) can be integrated in terms of Jacobian elliptic functions for the case of travelling-wave solutions. Writing A in the form

$$A(\xi,\tau) = \exp\{il(\xi - V\tau)\}f(\xi - C\tau) \tag{4.14}$$

shows that

$$B = (\alpha/C)f^2, \quad f'' + af^3 - bf = 0, \tag{4.15}$$

where

$$a = -\frac{\alpha}{\lambda C}, \quad b = \frac{C^2}{4\lambda^2}\left(1 - 2\frac{V}{C}\right), \quad l = \frac{C}{2\lambda}. \tag{4.16}$$

The solutions for the envelope function $f(X)$ are

(i) $$\left.\begin{array}{l} f(X) = A_0 \,\mathrm{dn}\,(\beta X|m), \quad a > 0, \quad \tfrac{1}{2} < b/aA_0^2 < 1, \\ \beta^2 = \tfrac{1}{2}aA_0^2, \quad m = 2\{1 - (b/aA_0^2)\}; \end{array}\right\} \tag{4.17a}$$

(ii) $$\left.\begin{array}{l} f(X) = A_0 \,\mathrm{cn}\,(\beta X|m), \quad a > 0, \quad -\infty < b/aA_0^2 < \tfrac{1}{2}, \\ \beta^2 = aA_0^2/2m, \quad m = \tfrac{1}{2}\{1 - (b/aA_0^2)\}^{-1}; \end{array}\right\} \tag{4.17b}$$

(iii) $$\left.\begin{array}{l} f(X) = A_0 \,\mathrm{sn}\,(\beta X|m), \quad a < 0, \quad 1 < b/aA_0^2 < \infty, \\ \beta^2 = -aA_0^2/2m, \quad m = \{(2b/aA_0^2) - 1\}^{-1}; \end{array}\right\} \tag{4.17c}$$

(iv) $$\left.\begin{array}{l} f(X) = A_0 \,\mathrm{cd}\,(\beta X|m), \quad a < 0, \quad -\infty < b/aA_0^2 < 1, \\ \beta^2 = -aA_0^2/(m+1), \quad m = (b/a)A_0^2/[2 - (b/a)A_0^2]; \end{array}\right\} \tag{4.17d}$$

(v) $$\left.\begin{array}{l} f(X) = A_0 \,\mathrm{nd}\,(\beta X|m), \quad a > 0, \quad 1 < b/aA_0^2 < \infty, \\ \beta^2 = aA_0^2/2(1-m), \quad m = (b/a)A_0^2 - 1/[(b/a)A_0^2 - 2]. \end{array}\right\} \tag{4.17e}$$

The gravity–capillary wave problem studied here has α, $\lambda > 0$, so that only solutions (iii) and (iv) are admissible. In the limit $m = 1$, solution (iii) becomes

$$f(X) = A_0 \tanh(\beta X) \tag{4.18}$$

and solution (iv) becomes

$$f(X) = A_0. \tag{4.19}$$

The first case is equivalent to the phase-jump solution to the nonlinear Schrödinger equation (Hasimoto & Ono 1972). Also, the general inverse-scattering-transform method developed by Ablowitz *et al.* (1974) can be used to obtain the complete solution but we do not pursue that here.

A further aspect of (4.12) which is worthy of discussion is the stability of the solution

$$A = A_0 \exp(-iB_0\tau), \quad B = B_0 \tag{4.20}$$

for a uniform wave train. Superimposing a small modulational perturbation of the form

$$A = A_0\{1 + a(\xi,\tau)\}\exp(-iB_0\tau), \quad B = B_0\{1 + b\{\xi,\tau\}, \tag{4.21}$$

where

$$a = a_+ \exp\{i(K\xi + \sigma\tau)\} + a_- \exp\{-i(K\xi + \sigma\tau)\}, \tag{4.22}$$

and linearizing, one obtains the eigenvalue relation

$$\sigma^3 - (K^2\lambda)^2\sigma + 2\alpha\lambda K^3|A_0|^2 = 0. \tag{4.23}$$

The wave train is unstable to disturbances satisfying the condition

$$3^{\frac{3}{2}}|\alpha|\,|A_0|^2 > \lambda^2 K^3. \tag{4.24}$$

There always exists a modulation wavelength long enough for instability. As was found for the case of self-modulation, it is expected that the envelope solution admitting solitons will play a major role in the dynamics of the interaction described by (4.12).

5. Concluding remarks

The dynamical evolution equations for weakly nonlinear capillary–gravity waves have been derived and used to study the stability of a Stokes wave train. It is found that there are several wavenumber bands within which a capillary–gravity wave train is unstable to modulational (side-band) perturbations. Within these unstable bands the wave motion is known to depend crucially on the existence of envelope solitons and a uniform wave train does not exist. In practice, however, detection of the side-band instability and soliton formation in the capillary regime may be offset by the rapid viscous dissipation occurring at high wavenumbers. The maximum growth rate for the side-band instability is $O(\epsilon^2 A_0^2|\nu|)$, which must be compared with the viscous decay, which is $O(\nu_* k^2)$, where ν_* is the kinematic viscosity of the fluid. $|\nu|$ becomes large in the neighbourhood of the resonant loci and therefore these are the wavenumber bands in which the modulation of capillary–gravity waves is most likely to be observed.

The existence of a resonant interaction between a shallow-water gravity wave and a capillary wave has emerged from the analysis and the equations describing this interaction have been derived. It is found that the resonant interaction occurs on a time scale $O(\epsilon^{-\frac{4}{3}})$ compared with $O(\epsilon^{-2})$ for self-modulation, where ϵ is a measure of the amplitude (nonlinearity) of the capillary–gravity wave. The dynamical characteristics of this resonant interaction are being studied further.

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Appendix

We present here the evolution equations describing the self-modulation of pure capillary waves ($g = 0$). The equations equivalent to (2.13) and (2.14) are

$$iA_\tau + \tfrac{1}{2}\omega'' A_{\xi\xi} + \frac{c_g}{2k} A_{\eta\eta} = \frac{k^4 T}{2\omega}(1-\sigma^2)\,\zeta^{(2,0)} A - \frac{k^4}{4\omega}\left\{1 + 2(1-\sigma^2)\left(3 - 2\frac{c_p}{c_g}\right)\right\}|A|^2 A, \tag{A 1}$$

$$c_g \zeta_\xi^{(2,0)} = h\{\Phi_{\xi\xi}^{(1,0)} + \Phi_{\eta\eta}^{(1,0)}\} + (2k^2\sigma/\omega)\,(|A|^2)_\xi \tag{A 2}$$

and

$$c_g \Phi_\xi^{(1,0)} = k^2(1-\sigma^2)\,|A|^2. \tag{A 3}$$

The dispersion relation is

$$\omega = k^{\frac{3}{2}}(\sigma T)^{\frac{1}{2}} \tag{A 4}$$

and the group speed is given by

$$c_g = (k^2 T/2\omega)\{3\sigma + kh(1-\sigma^2)\}. \tag{A 5}$$

Defining the variable $Q(\xi,\eta,\tau)$ as

$$Q = c_g\zeta^{(2,0)} - \frac{k}{c_g}\left\{kh(1-\sigma^2)+2\sigma\frac{c_g}{c_p}\right\}|A|^2, \tag{A 6}$$

one obtains the pair of equations

$$iA_\tau + \lambda A_{\xi\xi} + \mu A_{\eta\eta} = \nu_c |A|^2 A + \nu_2 QA \tag{A 7}$$

and

$$Q_{\xi\xi} = \kappa_2(|A|^2)_{\eta\eta}, \tag{A 8}$$

where λ and μ have the same definitions as in (2.17) and the remaining coefficients are given by

$$\left.\begin{aligned} \nu_c &= -\frac{k^4}{4\omega}\left\{1+6(1-\sigma^2)\left[\frac{\sigma+kh(1-\sigma^2)}{3\sigma+kh(1-\sigma^2)}\right]^2\right\}, \\ \nu_2 &= \frac{k^3T}{2c_pc_g}(1-\sigma^2), \quad \kappa_2 = \frac{k^2h}{c_g}(1-\sigma^2). \end{aligned}\right\} \tag{A 9}$$

In this case $\lambda > 0$ nd $\nu < 0$ for all kh, so that pure capillary waves are always modulationally unstable.

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